Flows and Duality Symmetries in N=1 Supersymmetric Gauge Theories

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We present more examples of dual N=1 SUSY gauge theories. This set of theories is connected by flows to both Seiberg's and Kutasov's dual theories. This provides a unifying picture of the various dual theories. We investigate the dual theories, their flat directions and mass perturbations.

1. Introduction

Duality symmetries are playing an increasing role in the understanding of the behavior of supersymmetric gauge theories at strong coupling. The electric–magnetic dualities relating strong and weak coupling make sense in the context of $N = 4$ supersymmetric gauge theories [1,2], where they are expected to be exact symmetries. They were generalized [3,4,5] to $N = 2$ theories and to some $N = 1$ theories [6] in the Coulomb phase, and enabled the computation of non–perturbative results in these theories. There is growing evidence [7,8] that these $SL(2, Z)$ duality symmetries appear in string theory as well.

A new type of duality symmetry, connecting $N = 1$ supersymmetric gauge theories with (generally) different gauge groups, was suggested by Seiberg in [9]. This duality relates a standard supersymmetric QCD (SQCD) theory with gauge group $SU(N_c)$ and N_f flavors of quarks, with another supersymmetric QCD theory, with gauge group $SU(N_f - N_c)$, N_f quark flavors and additional gauge–singlet fields with a non–trivial superpotential coupling the singlets to the dual quarks. The evidence for Seiberg's duality consists [9] of (i) the identification of the global anomalies and gauge–invariant operators in the chiral rings of the two theories, (ii) the possibility of flowing (by mass perturbations) between different pairs of dual theories and from them to known [10] effective descriptions of SQCD (for small values of N_f) in terms of bound states, and (iii) the identification of all flat directions of the two theories. These include in particular [11] a flat direction along which both theories flow to the same field theory in the IR. The generalization of this duality to the $SO(N_c)$ case (discussed in [9,12]) is related to the electric–magnetic duality, with the dual quarks interpreted as monopoles (or dyons) of the original theory. In the $SU(N_c)$ case no monopoles appear (semi–classically) in the spectrum, so that the connection of the new duality to the electric–magnetic duality is still obscure, though there may be some relations between the two [13]. For $N_f = N_c + 1$ the duality goes over to the description of the theory in terms of bound states, with the dual quarks becoming baryons.

Another duality of this type was suggested by Kutasov [14]. The original ("electric") theory in this duality is an $SU(N_c)$ gauge theory ($N_c > 2$), with N_f flavors of quarks Q_a^i and anti–quarks $\tilde{Q}_i^a$, and an additional adjoint field X, with a superpotential $W_{el} = \text{Tr} (X^3)$. The dual ("magnetic") theory is an $SU(2N_f - N_c)$ gauge theory, with N_f flavors of quarks q_i^a and anti–quarks $\tilde{q}_a^{\tilde{i}}$, an adjoint field Y, and singlet fields $M_{\tilde{i}}^i$ (identified with $Q^i \tilde{Q}_{\tilde{i}}$) and $N_{\tilde{i}}^i$ (identified with $Q^i X \tilde{Q}_{\tilde{i}}$). The superpotential in the "magnetic" theory [14] is given by $W_{mag} = M_{\tilde{i}}^i q_i Y \tilde{q}^{\tilde{i}} + N_{\tilde{i}}^i q_i \tilde{q}^{\tilde{i}} + \text{Tr} (Y^3)$. The evidence given in [14] for this

duality consists of the identification of the global anomalies and gauge–invariant operators in the two theories, and the possibility of flowing between different values of N_f by mass perturbations. As in Seiberg's duality, for $N_f = \frac{1}{2}(N_c+1)$ the duality becomes a description of the theory in terms of bound states, with the dual quarks becoming baryons. In other cases the interpretation of the dual quarks in terms of the original variables is not clear.

In this paper we present another duality transformation of this type, which can be reached from Kutasov's duality [14] by flowing along a flat direction. In addition to the fields described above, both theories in this duality have $\tilde{N}_f$ additional flavors of quarks Z_a^j and anti–quarks $\tilde{Z}_j^a$, which couple to the adjoint field by a coupling of the form $Z_a^j X_b^a \tilde{Z}_j^b$. The duality takes the gauge group $SU(N_c)$ to $SU(2N_f + \tilde{N}_f - N_c)$. We shall check the 't Hooft anomaly conditions and the identification of gauge–invariant bound states in the two theories, and analyze several mass perturbations of these theories. We shall also analyze several flat directions of these theories, finding all of them to be consistent with the duality. This analysis is non–trivial in view of the appearance of non–renormalizable terms in the superpotential. In view of the connection to Kutasov's duality, it provides more support to the duality conjecture of [14]. Our analysis will ignore the possible quantum corrections to the superpotential in both theories, whose analysis is left for future work. In most cases we will find a consistent picture without the quantum corrections. This indicates that in these cases the quantum corrections do not seem to be important. Among the theories we flow to we will find Seiberg's dual theories [9]. Thus, we relate the dualities of Seiberg and Kutasov. Moreover, some cases which will be considered seem to indicate a possible relation between the $N = 1$ duality and the duality of $N = 2$ theories [3-5].

In section 2 we describe the duality transformation, check the 't Hooft anomaly matching conditions and identify the gauge–invariant bound states. In section 3 we analyze flows generated by adding mass perturbations, and verify that the resulting theories are consistent with the duality. In section 4 we analyze several flat directions of the dual theories, finding them too to be consistent with the duality. In section 5 we analyze another flat direction which is more complicated. This flat direction relates our theories to Kutasov's dual theories [14]. We end in section 6 with a summary and conclusions.

2. The duality transformation

In this section we present a generalization of the known $N = 1$ SUSY duality [14]. The dual theories include, in addition to a field in the adjoint of the gauge group, two types of

quarks and anti–quarks (each in the fundamental and anti–fundamental representations of the gauge group). One type of quark couples to the adjoint field and the other does not. We will have N_f quarks without, and $\tilde{N}_f$ quarks with coupling to the adjoint field. The duality will take an $SU(N_c)$ gauge theory to an $SU(2N_f + \tilde{N}_f - N_c)$ gauge theory. The original ("electric") theory will be asymptotically free whenever $N_f + \tilde{N}_f < 2N_c$, and the dual ("magnetic") theory will be asymptotically free for $3N_f + \tilde{N}_f > 2N_c$.

2.1. The "electric" theory

The "electric" theory of the duality we analyze includes two types of quarks. There are N_f flavors of quarks Q_a^i and anti–quarks $\tilde{Q}_{\tilde{i}}^a$, with $a = 1, \cdots, N_c$ and $i, \tilde{i} = 1, \cdots, N_f$, and $\tilde{N}_f$ flavors of quarks Z_a^j and anti–quarks $\tilde{Z}_j^a$, with $a = 1, \cdots, N_c$ and $j = 1, \cdots, \tilde{N}_f$. There is also a field in the adjoint representation of the gauge group which will be denoted by X. We take the superpotential of the "electric" theory to be, at the classical level,

$$W_{el} = \text{Tr } (X^3) + 3 Z_a^j X_b^a \tilde{Z}_j^b. \tag{2.1}$$

The superpotential breaks the global symmetry classically to $U(N_f) \times U(N_f) \times U(\tilde{N}_f) \times U(1)_R$, and the instantons break one of the $U(1)$ factors, so that the quantum global symmetry of this theory is

$$SU(N_f) \times SU(N_f) \times SU(\tilde{N}_f) \times U(1)_B \times U(1)_Z \times U(1)_R \tag{2.2}$$

where we choose $U(1)_B$ and $U(1)_Z$ to be two arbitrary $U(1)$ symmetries orthogonal to the $U(1)_R$ symmetry.

The quantum numbers of all fields under the local symmetry $SU(N_c)$ and under the global symmetry (2.2) may be easily computed, and they are summarized in the following table:

		$SU(N_f)$	$SU(N_f)$	$SU(\tilde{N}_f)$	$U(1)_B$	$U(1)_Z$	$U(1)_R$
Q_a^i	N_c	N_f	1	1	$\frac{1}{N_c}$	$-\frac{N_f - N_c}{N_c}$	$1 + \frac{1}{3}\frac{\tilde{N}_f - 2N_c}{N_f}$
$\tilde{Q}_{\tilde{i}}^a$	$\overline{N_c}$	1	$\overline{N_f}$	1	$-\frac{1}{N_c}$	$\frac{N_f - N_c}{N_c}$	$1 + \frac{1}{3}\frac{\tilde{N}_f - 2N_c}{N_f}$
Z_a^j	N_c	1	1	$\tilde{N}_f$	$\frac{1}{N_c}$	$-\frac{(N_c + \tilde{N}_f)N_f}{N_c \tilde{N}_f}$	$\frac{2}{3}$
$\tilde{Z}_j^a$	$\overline{N_c}$	1	1	$\overline{\tilde{N}_f}$	$-\frac{1}{N_c}$	$\frac{(N_c + \tilde{N}_f)N_f}{N_c \tilde{N}_f}$	$\frac{2}{3}$
X_b^a	$(N_c^2 - 1)$	1	1	1	0	0	$\frac{2}{3}$
W_α	$(N_c^2 - 1)$	1	1	1	0	0	1

where the R charge always pertains to the lowest component of the superfield.

2.2. The "magnetic" theory

The "magnetic" theory has a field content similar to that of the "electric" theory, for gauge group $SU(2N_f+\tilde{N}_f-N_c)$, with additional gauge singlet fields. There are N_f flavors of quarks q_i^a and $\tilde{q}_a^{\tilde{i}}$, $\tilde{N}_f$ flavors of quarks z_j^a and $\tilde{z}_a^j$ and an adjoint field Y. Note that the non–abelian global representations of the "magnetic" quarks are conjugate to those of the "electric" quarks. In addition to these there are gauge singlet fields, which will be identified with mesons of the "electric" theory. These are $M_{\tilde{i}}^i$, $N_{\tilde{i}}^i$, $M_{\tilde{i}}^j$ and M_j^i, where i is an index of the first $SU(N_f)$ factor of the global symmetry, $\tilde{i}$ is an index of the second $SU(N_f)$ factor of the global symmetry, and j is an index of $SU(\tilde{N}_f)$. The global symmetry group is the same as (2.2). The classical superpotential of the "magnetic" theory is

$$W_{mag} = \mathrm{Tr}\ (Y^3) + 3z_j^a Y_a^b \tilde{z}_b^j + M_{\tilde{i}}^i q_i^a Y_a^b \tilde{q}_b^{\tilde{i}} + N_{\tilde{i}}^i q_i^a \tilde{q}_a^{\tilde{i}} + M_j^i \tilde{z}_a^j q_i^a + M_{\tilde{i}}^j z_j^a \tilde{q}_a^{\tilde{i}}. \qquad (2.3)$$

The non–renormalizable term in (2.3) is analogous to the one in [14], and can be relevant (as in [14]), due to non–perturbative effects, when the "magnetic" theory is strongly coupled. We will see in section 5 how to deal with this term when analyzing perturbations and flat directions of the theory.

The local and global symmetry charges of the fields may easily be computed. Making a choice of the $U(1)$ symmetries that will fit with the choice we made in the "electric" theory, we find them to be :

q_i^a	$2N_f+\tilde{N}_f-N_c$	($\overline{N_f}$,	1,	1,	$\frac{1}{2N_f+\tilde{N}_f-N_c}$,	-1,	$\frac{2N_c-N_f-\tilde{N}_f}{3N_f}$)
$\tilde{q}_a^{\tilde{i}}$	$\overline{2N_f+\tilde{N}_f-N_c}$	(1,	N_f,	1,	$-\frac{1}{2N_f+\tilde{N}_f-N_c}$,	1,	$\frac{2N_c-N_f-\tilde{N}_f}{3N_f}$)
$M_{\tilde{i}}^i$	1	(N_f,	$\overline{N_f}$,	1,	0,	0,	$\frac{6N_f+2\tilde{N}_f-4N_c}{3N_f}$)
$N_{\tilde{i}}^i$	1	(N_f,	$\overline{N_f}$,	1,	0,	0,	$\frac{8N_f+2\tilde{N}_f-4N_c}{3N_f}$)
$M_{\tilde{i}}^j$	1	(1,	$\overline{N_f}$,	$\tilde{N}_f$,	0,	$-\frac{N_f+\tilde{N}_f}{\tilde{N}_f}$,	$\frac{5N_f+\tilde{N}_f-2N_c}{3N_f}$)
M_j^i	1	(N_f,	1,	$\overline{\tilde{N}_f}$,	0,	$\frac{N_f+\tilde{N}_f}{\tilde{N}_f}$,	$\frac{5N_f+\tilde{N}_f-2N_c}{3N_f}$)
z_j^a	$2N_f+\tilde{N}_f-N_c$	(1,	1,	$\overline{\tilde{N}_f}$,	$\frac{1}{2N_f+\tilde{N}_f-N_c}$,	$\frac{N_f}{\tilde{N}_f}$,	$\frac{2}{3}$)
$\tilde{z}_a^j$	$\overline{2N_f+\tilde{N}_f-N_c}$	(1,	1,	$\tilde{N}_f$,	$-\frac{1}{2N_f+\tilde{N}_f-N_c}$,	$-\frac{N_f}{\tilde{N}_f}$,	$\frac{2}{3}$)
Y	$((2N_f+\tilde{N}_f-N_c)^2-1)$	(1,	1,	1,	0,	0,	$\frac{2}{3}$)
W_α	$((2N_f+\tilde{N}_f-N_c)^2-1)$	(1,	1,	1,	0,	0,	1).

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2.3. Comparing the two theories

The first things which one should check in verifying a duality symmetry are the 't Hooft anomaly matching conditions and the identification of the gauge–invariant operators in the chiral ring, which we will perform in this section. In later sections we will compare the mass perturbations of the two theories and several of their flat directions as well. One should also check that by performing the duality transformation twice one returns to the original theory. This works in our case in the same way as in the cases described in [9] and in [14].

The global anomalies in both theories may easily be calculated to be :

$$
\begin{array}{ll}
SU(N_f)^3 & N_c d^{(3)}(N_f) \\[4pt]
SU(N_f)^2 U(1)_B & d^{(2)}(N_f) \\[4pt]
SU(N_f)^2 U(1)_R & \dfrac{N_c(\tilde{N}_f - 2N_c)}{3N_f} d^{(2)}(N_f) \\[6pt]
SU(\tilde{N}_f)^2 U(1)_R & -\dfrac{2}{3} N_c d^{(2)}(\tilde{N}_f) \\[6pt]
SU(N_f)^2 U(1)_Z & (N_c - N_f) d^{(2)}(N_f) \\[6pt]
U(1)_R & -\dfrac{2}{3}(N_c^2 + 1) \\[6pt]
U(1)_R^3 & \dfrac{1}{27}(26N_c^2 - 2\tilde{N}_f N_c - 26) + \dfrac{2N_c}{27N_f^2}(\tilde{N}_f - 2N_c)^3 \\[6pt]
U(1)_B^2 U(1)_R & -\dfrac{4}{3} \\[6pt]
U(1)_Z^2 U(1)_R & -\dfrac{4}{3} N_c^2 - \dfrac{2}{3\tilde{N}_f} N_c N_f^2 + \dfrac{2}{3} N_c \tilde{N}_f - \dfrac{8}{3} N_f^2 - \dfrac{4}{3} N_f \tilde{N}_f + \dfrac{8}{3} N_f N_c
\end{array}
\tag{2.4}
$$

with all other anomalies vanishing trivially. Hence, the anomaly matching conditions indeed hold.

Next, let us compare the gauge–invariant operators, starting with the meson–like operators in the two theories. The "normal" mesons of the "electric" theory, which include at least one of Q or $\tilde{Q}$, may all be identified with gauge singlets of the "magnetic" theory :

$$
M_{\tilde{i}}^{i} \sim Q_a^i \tilde{Q}_{\tilde{i}}^a ; \qquad M_{\tilde{i}}^{j} \sim Z_a^j \tilde{Q}_{\tilde{i}}^a ; \qquad M_j^i \sim Q_a^i \tilde{Z}_j^a .
\tag{2.5}
$$

The mesons made out of a pair of Z-quark fields reside in the adjoint and singlet representations of the $SU(\tilde{N}_f)$ flavor group. Naively, all of them may be identified with the corresponding mesons in the "magnetic" theory, $Z^j \tilde{Z}_k \sim z_k \tilde{z}^j$, but we shall see that this

is only true for the adjoint mesons and not for the singlet. The other "normal" mesons of the "magnetic" theory are not in the chiral ring due to the equations of motion associated with the superpotential (2.3).

There are also generalized mesons which include the adjoint field. In the "electric" theory, these are the operators $Q_a^i X_b^a \tilde{Q}_i^b$, which may be identified with N_i^i in the "magnetic" theory. All other generalized mesons, in both theories, which include Z-quarks or more than one power of the adjoint field, are not in the chiral ring. This can be verified by using the equations of motion associated with the superpotentials of the two theories, which allow us to set them to zero or to combinations of other singlet fields.

There is one more relatively simple operator in each of the chiral rings of these theories. It is Tr (X^2) in the "electric" theory, and Tr (Y^2) in the "magnetic" theory. Superficially these two operators should be identified, but in fact there is another operator with the same quantum numbers, which is the singlet meson $Z_a^j \tilde{Z}_j^a$ in the "electric" theory and $z_j^a \tilde{z}_a^j$ in the "magnetic" theory. Thus, there can be a mixing between the two operators, and the most general possible identification is of the form

$$
\begin{aligned}
\text{Tr } (Y^2) &\sim a\text{Tr } (X^2) + b Z_a^j \tilde{Z}_j^a \\
z_j^a \tilde{z}_a^j &\sim c\text{Tr } (X^2) + d Z_a^j \tilde{Z}_j^a.
\end{aligned}
\tag{2.6}
$$

We shall see that all the coefficients a, b, c and d are in fact non–zero.

The equations of motion resulting from the superpotential also force the baryon–like operators in the chiral ring of the two theories to be of the form :

$$
\begin{aligned}
B^{[i_1,\cdots,i_k][i_{k+1},\cdots,i_n][j_1,\cdots,j_m]} = \epsilon^{\alpha_1,\cdots,\alpha_{N_c}} X_{\alpha_1}^{\beta_1} \cdots X_{\alpha_k}^{\beta_k} Q_{\beta_1}^{i_1} \cdots Q_{\beta_k}^{i_k} Q_{\alpha_{k+1}}^{i_{k+1}} \cdots Q_{\alpha_n}^{i_n} \cdot \\
Z_{\alpha_{n+1}}^{j_1} \cdots Z_{\alpha_{n+m}}^{j_m}
\end{aligned}
\tag{2.7}
$$

with $n + m = N_c$. It can easily be checked that these baryons in the "electric" theory have exactly the same global symmetry numbers as the baryons in the "magnetic" theory (constructed in the same way from Y's, q's and z's) with $\tilde{k} = N_f - n + k$, $\tilde{n} = 2N_f - n$ and $\tilde{m} = \tilde{N}_f - m$. Hence, all these operators may be identified.

More general hadronic operators involving quarks and anti–quarks also appear in these theories [14]. These are, in the "electric" theory, of the form

$$
\begin{aligned}
C_{[\tilde{i}_1,\cdots,\tilde{i}_m][k_1,\cdots,k_n]}^{[i_1,\cdots,i_k][j_1,\cdots,j_l]} = \epsilon^{\alpha_1,\cdots,\alpha_{N_c}} \epsilon_{\beta_1,\cdots,\beta_{N_c}} Q_{\alpha_1}^{i_1} \cdots Q_{\alpha_k}^{i_k} Z_{\alpha_{k+1}}^{j_1} \cdots Z_{\alpha_{k+l}}^{j_l} \tilde{Q}_{\tilde{i}_1}^{\beta_1} \cdots \tilde{Q}_{\tilde{i}_m}^{\beta_m} \cdot \\
\tilde{Z}_{k_1}^{\beta_{m+1}} \cdots \tilde{Z}_{k_n}^{\beta_{m+n}} X_{\alpha_{k+l+1}}^{\beta_{m+n+1}} \cdots X_{\alpha_{N_c}}^{\beta_{N_c}}
\end{aligned}
\tag{2.8}
$$

with $k + l = m + n$. Similar operators exist in the "magnetic" theory. It can easily be checked that these operators may also be identified between the two theories, with $\tilde{k} = N_f - k$, $\tilde{m} = N_f - m$, $\tilde{l} = n$ and $\tilde{n} = l$.

The operators we analyzed in this section seem to be the only independent gauge–invariant operators in the chiral ring of the two theories (actually not all of the B and C hadrons are independent operators in the chiral ring), and we have established that they are identical in the two theories.

2.4. Special cases of the duality

As described above, the duality holds for all values of N_f, N_c and $\tilde{N}_f$, as long as $2N_f + \tilde{N}_f > N_c$. The only difference for $\tilde{N}_f = 0$ or $N_f = 0$ is that some of the $U(1)$ symmetries disappear. In the case of $\tilde{N}_f = 0$ one $U(1)$ symmetry disappears, and the duality goes over to Kutasov's duality [14]. Hence, the evidence we give in the next sections for the duality may be used as further support for Kutasov's duality. Of course, the quantum corrections may destroy the duality in some of these cases, if they remove the origin of moduli space from the quantum moduli space. This happens for instance in SQCD when $N_f \leq N_c$ [10]. However, we see no a–priori reason for this to occur in our theories, and in particular we have a solution for the 't Hooft anomaly conditions whenever $2N_f + \tilde{N}_f > N_c$.

For $N_f = 0$ two of the $U(1)$ symmetries disappear (for general $\tilde{N}_f$), and in particular we no longer have a $U(1)_R$ symmetry. In both theories we are left with only the adjoint field and with the Z-quarks, which have an R-charge of $2/3$ and an interaction with the adjoint field. The duality in this case takes the gauge group $SU(N_c)$ to $SU(\tilde{N}_f - N_c)$, like the duality described by Seiberg [9] without the adjoint field. The "electric" theory is now asymptotically free for $\tilde{N}_f < 2N_c$, and the "magnetic" theory is asymptotically free for $\tilde{N}_f > 2N_c$. Therefore, we do not have an IR fixed point for any value of $\tilde{N}_f$. This is consistent with the fact that we have no $U(1)_R$ symmetry in this case. For $\tilde{N}_f = 2N_c$ the $U(1)_R$ symmetry remains unbroken, and we find that the original theory is equal to the dual theory (except for the flavor representations of the quarks). In this case the theory may be exactly conformal, and the duality we analyze may be related to the $N = 2$ duality discovered by Seiberg and Witten [3-5]. Note that the coupling of the quarks to the adjoint fields is the same as in the $N = 2$ theory. The main difference seems to be the existence of the Tr (X^3) term in the superpotential, which explicitly breaks the $N = 2$ symmetry (and the R charge associated with it). It is possible that by turning on a perturbation

proportional to this operator in the $N = 2$ theory, one may flow from the $N = 2$ duality (with $N_f = 2N_c$ and $N_c > 2$) to our duality for $N_f = 0$ and $\tilde{N}_f = 2N_c$. From there, one can flow to other values of $\tilde{N}_f$ as described in the next sections. Unfortunately, we have not identified any flows which increase the value of N_f. We, therefore, still do not know how to flow to all cases from this one. However, one can flow from the case of general N_f to the case of $N_f = 0$. Hence, the understanding of the case of $N_f = 0$ may help in understanding the general case as well.

3. Mass perturbations of the dual theories

Another comparison which one can make between the two theories concerns their behavior under mass perturbations. In this section we will analyze mass perturbations of the dual theories. We will ignore the possible quantum corrections to the superpotential whenever this is possible. In most cases we will see that this indeed gives a flow consistent with the duality. However, it seems clear that, as in Seiberg's duality [9], whenever one of the gauge groups is completely broken the quantum corrections become important. Therefore, all the flows we describe here, and in the next section when we analyze the flow along flat directions, are presumably relevant only when neither gauge group is completely broken.

3.1. Quark mass perturbations of the dual theories

When we add a mass to an "electric" Q-quark, i.e. a term $mM_{N_f}^{N_f}$ in the superpotential, the behavior is the same as in Kutasov's original theory [14]. In the "electric" theory the quarks Q^{N_f} and $\tilde{Q}_{N_f}$ become massive, reducing N_f by one. In the "magnetic" theory the equation of motion of $M_{N_f}^{N_f}$ forces us to give VEVs to q_{N_f}, $\tilde{q}^{N_f}$ and Y, reducing N_f by one and the number of colors by two. The resulting theories are again dual under the duality transformation.

Adding a mass to the Z-quarks is more subtle, due to the identification (2.6), and will be discussed later. However, it is relatively simple to perturb the theory by a mass operator of the form $mQ^{N_f}\tilde{Z}_{\tilde{N}_f} \sim mM_{\tilde{N}_f}^{N_f}$. In the "electric" theory, this gives a mass to Q^{N_f} and to $\tilde{Z}_{\tilde{N}_f}$, and leaves $Z^{\tilde{N}_f}$ without a coupling to the adjoint field, i.e. it becomes a regular Q-quark. Thus, we reduce $\tilde{N}_f$ by one, leaving N_c and N_f as they were. In the "magnetic" theory, the equation of motion of $M_{\tilde{N}_f}^{N_f}$ forces $\tilde{z}^{\tilde{N}_f}$ and q_{N_f} to get VEVs, breaking the "magnetic" gauge symmetry to $SU(2N_f + \tilde{N}_f - N_c - 1)$. Let us choose the

components that get non–zero VEVs to be $q^1_{N_f}$ and $\tilde{z}^{\tilde{N}_f}_1$. Then, $\tilde{z}^{\tilde{N}_f}$ and q_{N_f} are swallowed by the Higgs mechanism, and the fields $z^a_{\tilde{N}_f}$ and Y^1_a get a mass from the superpotential (as well as some other fields). When integrating out the massive fields we find that Y^a_1 now couples just like a regular q-quark. Hence, N_f does not change while $\tilde{N}_f$ is reduced by one. This is exactly the dual of the result we found in the "electric" theory. Thus, this perturbation preserves the duality.

3.2. Mass perturbations of the Z-quarks

Since the identification of the singlet Z-meson in (2.6) is complicated, let us analyze first the Z mass terms which are in the adjoint representation of the flavor group. An example of such a term is $m(Z^1\tilde{Z}_1 - Z^2\tilde{Z}_2) \sim m(z_1\tilde{z}^1 - z_2\tilde{z}^2)$. Superficially this gives a mass to two Z-quarks in both theories, reducing $\tilde{N}_f$ by two in both theories. This is obviously inconsistent with the duality. However, because of the coupling of the Z-quarks to the adjoint field, we must be careful in interpreting m as the actual "physical" mass of the quarks. There is a vacuum in which X (Y) does not get a VEV, and then m is indeed the mass of the quarks. However, there could also be some other vacua. There is a-priori no reason to identify the vacuum in which the VEVs of all fields are zero in the "electric" theory with the same vacuum of the "magnetic" theory. The only demand is to identify all the gauge–invariant operators in the corresponding theories. Let us try, therefore, to find a vacuum for which X gets a VEV in the "electric" theory, in the theory with finite m (the analysis of the "magnetic" theory is analogous). Note that this theory does not have an R symmetry, and, therefore, in the IR m must flow either to zero or to infinity. One can easily find that the following VEVs satisfy all the F-term and D-term constraints of the perturbed theory :

$$\langle X^1_1 \rangle = -\frac{1}{3}m; \langle X^2_2 \rangle = \frac{1}{3}m; \langle Z^1_1 \rangle = \frac{1}{3}m; \langle \tilde{Z}^1_1 \rangle = -\frac{1}{3}m; \langle Z^2_2 \rangle = \frac{1}{3}m; \langle \tilde{Z}^2_2 \rangle = -\frac{1}{3}m \quad (3.1)$$

with all other VEVs vanishing. Along this flat direction we find that the gauge group is broken to $SU(N_c - 2)$, and all components of $Z^1, \tilde{Z}_1, Z^2$ and $\tilde{Z}_2$ become massive, either by contributions from the superpotential or through the Higgs mechanism. Hence, along this flat direction we flow to a theory with both $\tilde{N}_f$ and the number of colors reduced by two. This is dual to the theory we found for zero VEVs, which had $\tilde{N}_f$ reduced by two without a change in the number of colors. Thus, when we include all flat directions, this mass perturbation preserves the duality symmetry.

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Let us now discuss the perturbation by the mass operator of one of the z-quarks in the "magnetic" theory. In the "magnetic" theory, this perturbation is $mz_1\tilde{z}^1$ which we may write as

$$m((\tilde{N}_f - 1)z_1\tilde{z}^1 - z_2\tilde{z}^2 - \cdots - z_{\tilde{N}_f}\tilde{z}^{\tilde{N}_f} + z_j\tilde{z}^j)/\tilde{N}_f \qquad (3.2)$$

dividing it into components which are in the adjoint and singlet representations of the flavor group $SU(\tilde{N}_f)$. Now, according to (2.6), we should identify this operator in the "electric" theory with

$$m((\tilde{N}_f - 1)Z^1\tilde{Z}_1 - Z^2\tilde{Z}_2 - \cdots - Z^{\tilde{N}_f}\tilde{Z}_{\tilde{N}_f} + c\text{Tr}\,(X^2) + dZ^j\tilde{Z}_j)/\tilde{N}_f \qquad (3.3)$$

which equals

$$m((\tilde{N}_f + d - 1)Z^1\tilde{Z}_1 + (d - 1)(Z^2\tilde{Z}_2 + \cdots + Z^{\tilde{N}_f}\tilde{Z}_{\tilde{N}_f}) + c\text{Tr}\,(X^2))/\tilde{N}_f. \qquad (3.4)$$

As in the previous section, it is not obvious that m is actually the "physical" mass of z_1 and $\tilde{z}^1$, because of the $z_1Y\tilde{z}^1$ coupling in the superpotential. However, there is certainly a flat direction of the "magnetic" theory for which the VEV of Y vanishes, and then m is indeed the "physical" mass of these fields. Along this flat direction, we find in the "magnetic" theory that $\tilde{N}_f$ decreases by one with no change in N_f and in the gauge group. Thus, we should be able to find in the "electric" theory the dual of this result, i.e. a flat direction along which N_f is unchanged, and both $\tilde{N}_f$ and N_c decrease by one. By demanding the existence of such a flat direction we will be able to fix the values of c and d in equation (2.6).

The most general VEVs we can have in the "electric" theory which will satisfy the D-term equations, break the color group to $SU(N_c - 1)$, break the Z-flavor group to $SU(\tilde{N}_f - 1)$ and will not affect the Q-flavor group, are of the form (up to global and local transformations)

$$\langle X_1^1 \rangle = A$$
$$\langle X_j^j \rangle = B \qquad (j = 2, \cdots, N_c) \qquad (3.5)$$
$$\langle Z_1^1 \rangle = \langle \tilde{Z}_1^1 \rangle = F$$

with some constants A, B and F. All other VEVs vanish. We can obtain several equations relating A, B, F, c and d. First of all, since X must be traceless, we find that

$$A + (N_c - 1)B = 0. \qquad (3.6)$$

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Next, we demand that these VEVs be solutions of the equations of motion resulting from the perturbed superpotential. The equation of motion of X in the perturbed theory implies that the matrix $3X^2 + \frac{2mc}{\tilde{N}_f} X + 3Z^j \tilde{Z}_j$ must be proportional to the identity matrix. This gives an additional equation,

$$3B^2 + \frac{2mc}{\tilde{N}_f} B = 3A^2 + \frac{2mc}{\tilde{N}_f} A + 3F^2. \tag{3.7}$$

The equations of motion of Z^1 and $\tilde{Z}_1$ lead to another equation,

$$3A + m\frac{\tilde{N}_f + d - 1}{\tilde{N}_f} = 0. \tag{3.8}$$

We need two more equations to determine all the constants, and these will be obtained by demanding that all the fields Z^j, $\tilde{Z}_j$ for $j = 2, \cdots, \tilde{N}_f$ and X_k^j for $j, k = 2, \cdots, N_c$ remain massless along this flat direction. (Recall that we want to flow to the "electric" theory with $\tilde{N}_f - 1$ massless flavors of Z-quarks and $N_c - 1$ colors). By demanding that $\tilde{N}_f - 1$ flavors of Z-quarks remain massless we find

$$3B + m\frac{d - 1}{\tilde{N}_f} = 0 \tag{3.9}$$

and by demanding that the $SU(N_c - 1)$ adjoint components of X remain massless we find

$$3B + \frac{cm}{\tilde{N}_f} = 0. \tag{3.10}$$

The above five equations have a unique solution for A, B, F, c and d. Thus, we have found the values of c and d in (2.6):

$$c = -\frac{\tilde{N}_f}{N_c}; \qquad d = 1 - \frac{\tilde{N}_f}{N_c}. \tag{3.11}$$

We have also found, in the "electric" theory, a flat direction along which the theory flows to the dual of the theory that we have found when we considered the "magnetic" theory. The duality is, therefore, consistent with this mass perturbation.

We can find the other constants in equation (2.6) by using the inverse equation

$$\begin{aligned} \text{Tr } (X^2) &\sim \tilde{a}\text{Tr } (Y^2) + \tilde{b}z_j^a \tilde{z}_a^j \\ Z_a^j \tilde{Z}_j^a &\sim \tilde{c}\text{Tr } (Y^2) + \tilde{d}z_j^a \tilde{z}_a^j. \end{aligned} \tag{3.12}$$

By repeating the previous analysis with a perturbation proportional to $Z^1 \tilde{Z}_1$, we can easily find that

$$\tilde{c} = -\frac{\tilde{N}_f}{2N_f + \tilde{N}_f - N_c}; \qquad \tilde{d} = 1 - \frac{\tilde{N}_f}{2N_f + \tilde{N}_f - N_c}. \tag{3.13}$$

The consistency of (2.6) and (3.12) requires that $c\tilde{a} + d\tilde{c} = a\tilde{c} + c\tilde{d} = b\tilde{a} + d\tilde{b} = a\tilde{b} + b\tilde{d} = 0$ and that $c\tilde{b} + d\tilde{d} = b\tilde{c} + d\tilde{d} = a\tilde{a} + b\tilde{c} = a\tilde{a} + c\tilde{b} = 1$. The solution to all these equations is

$$a = \frac{N_c - 2N_f}{N_c}; \qquad b = -\frac{2N_f}{N_c}; \qquad \tilde{a} = \frac{\tilde{N}_f - N_c}{2N_f + \tilde{N}_f - N_c}; \qquad \tilde{b} = -\frac{2N_f}{2N_f + \tilde{N}_f - N_c}. \tag{3.14}$$

The identification of the operators with these values of the constants is the only one which is consistent with the duality, and we will assume it to hold.

3.3. Mass perturbations of the adjoint fields

Next, we consider the behavior of the theories under a mass perturbation associated with a mass operator for the adjoint fields. We shall start with the case $\tilde{N}_f = 0$, and later analyze the differences which arise in the case $\tilde{N}_f > 0$. When $\tilde{N}_f = 0$, the identification of the mass operator for the adjoint fields is necessarily $m\text{Tr}\,(X^2) \propto m\text{Tr}\,(Y^2)$. As in the previous discussions, we ignore the quantum corrections to the superpotential. However, for this perturbation we will find that (at least) in some cases they do play an important role.

When we add this operator to the "electric" theory, the equation of motion of X implies that $3X^2 + 2mX$ (where X here is the VEV of the field X) must be proportional to the identity matrix. One trivial solution of this equation of motion is $X = 0$. In this case we will have no breaking of the gauge symmetry, and m will be the "physical" mass of the field X. However, there are also other solutions. Since X obeys $\text{Tr}\,(X) = 0$, it can easily be seen that the most general solution (up to gauge transformations) for which no other fields get VEVs is $X^i_i = A, i = 1, \cdots, k$ and $X^j_j = B, j = k+1, \cdots, N_c$, where $0 < k < N_c/2$, $A = \frac{2}{3}m\frac{N_c - k}{2k - N_c}$ and $B = \frac{2}{3}m\frac{k}{N_c - 2k}$. The previous case of $X = 0$ may be obtained from this one by taking $k = 0$. To analyze the spectrum along this flat direction, let us write the matrix X in the form

$$X = \begin{pmatrix} AI + X_1 & X_2 \\ X_3 & BI + X_4 \end{pmatrix} \tag{3.15}$$

where I is the identity matrix and we divide the rows/columns of X into groups of size $(k, N_c - k)$. By examining the superpotential, we can see that the fields corresponding

to the matrices X_1 and X_4 become massive, while those corresponding to X_2 and X_3 remain massless and are swallowed by the Higgs mechanism. The gauge group breaks into $SU(k) \times SU(N_c - k) \times U(1)$. The quarks and anti–quarks are divided into two groups. The first one includes those which are charged under $SU(k)$ and singlets of $SU(N_c - k)$, while the other includes those which are charged under $SU(N_c - k)$ and singlets of $SU(k)$. When we integrate out the massive fields, the generated superpotential of the "electric" theory along this flat direction turns out to be zero. Thus, we flow to a theory containing two copies of supersymmetric QCD, one with N_f flavors and gauge group $SU(k)$, and the other with N_f flavors and gauge group $SU(N_c - k)$. There is an additional local $U(1)$ symmetry whose charge is proportional to the baryon number in each of the two theories.

Let us now analyze the same perturbation in the "magnetic" theory. The analysis of the possible VEVs of Y is similar to the one we have performed in the "electric" theory. The possible breakings of the gauge group are to $SU(\tilde{k}) \times SU(2N_f - N_c - \tilde{k}) \times U(1)$ with $0 \leq \tilde{k} < N_f - N_c/2$. The only difference is that in the "magnetic" theory there is also a coupling of the form $MqY\tilde{q}$, and we must understand what a large VEV for Y does to this non–renormalizable term. We postpone the analysis of this issue to section 5.2 and mention here only the results. We find that a large VEV for Y causes, after integrating out the massive fields, this term to go over to a term of the form $M^i_{\tilde{i}} q^a_i \tilde{q}^{\tilde{i}}_a$, plus non–renormalizable terms which appear to be irrelevant. The constant in front of this term depends on the VEV of Y, and thus we will find a different constant for the fields $q^a \tilde{q}_a$ with $a = 1, \cdots, \tilde{k}$ and for the fields $q^a \tilde{q}_a$ with $a = \tilde{k} + 1, \cdots, 2N_f - N_c$. We can then add this term to the original $Nq\tilde{q}$ term in the superpotential. We find that we have two independent meson fields in the $(N_f, \overline{N_f})$ representation of the $SU(N_f) \times SU(N_f)$ flavor group. One of the mesons (a linear combination of M and N) couples only to the first $\tilde{k}$ colors of quarks while the other couples only to the last $2N_f - N_c - \tilde{k}$ colors of quarks. As in the "electric" theory, the fields Y_2 and Y_3 also remain massless, and are swallowed by the Higgs mechanism. We thus flow to two copies of Seiberg's "magnetic" theory [9], one with N_f flavors and a gauge group $SU(\tilde{k})$ and the other with N_f flavors and a gauge group $SU(2N_f - N_c - \tilde{k})$. As in the "electric" theory, there is also a local $U(1)$ whose charge is proportional to the baryon number in each of the two theories.

Now that we have found the resulting theories in both cases, we should see how they are dual to each other. This means that we should match the flat directions which we have found, along which (when adding the mass perturbation) one can flow, between the two theories. Let us start with the case of $N_f \geq N_c$ and $0 < k < N_c/2$. This condition means

that in the "electric" theory we have $N_f > k$ and $N_f > N_c - k$. From Seiberg's results [9] we know that the theories of N_f flavors and k colors and of N_f flavors and $N_c - k$ colors each have a dual IR description. This description includes N_f flavors, $N_f - k$ colors and additional meson–like singlet fields for the first theory, and N_f flavors, $N_f - N_c + k$ colors and additional meson–like singlet fields for the second theory. (Strictly speaking, Seiberg's results regarding duality apply only for $N_f > N_c + 1$, but we can use them also for $N_f = N_c + 1$ since then they lead to a dual description in terms of mesons and baryons with no gauge symmetry [10]). This is exactly the theory we find when going along the flat direction of $\tilde{k} = N_f - N_c + k$ in the "magnetic" theory. Thus, along these flat directions we find that our duality goes over to two copies of Seiberg's duality [9], with an additional local $U(1)$ symmetry coupling to the baryon number of the respective "electric" and "magnetic" theories. The same analysis applies to the case of $N_f \leq N_c$ and $0 < \tilde{k} < N_f - N_c/2$ when we interchange the roles of the "electric" and "magnetic" theories.

We still need to understand the cases of $k = 0$ and $\tilde{k} = 0$, the cases of $0 < \tilde{k} < N_f - N_c$ when $N_f > N_c$ and the cases of $0 < k < N_c - N_f$ when $N_f < N_c$, for which we did not find dual theories in the discussion above. These cases all lead, at least in one of the theories (when identifying $\tilde{k} = N_f - N_c + k$), to a SQCD theory in which the number of quark flavors is smaller or equal to the number of colors. For these theories it is known that the quantum effects are important, and the quantum moduli space is not equal to the classical moduli space [10]. We will assume that the quantum effects along these flat directions in our theories are the same as in the corresponding SQCD theories [10]. We shall see that this assumption is consistent with the duality.

Let us start with the case of $k = 0$ for $N_f > N_c$. According to the above discussion, this should be dual to the case of $\tilde{k} = N_f - N_c$, which gives rise to a gauge theory of $SU(N_f) \times SU(N_f - N_c) \times U(1)$ in the "magnetic" theory. In the "electric" theory we get just SQCD with N_f flavors and N_c colors, for which we have [9] a dual description in terms of a "magnetic" theory based on the gauge group $SU(N_f - N_c)$. Thus, we need to understand the "extra" $SU(N_f)$ gauge theory with N_f flavors which we obtain in the "magnetic" theory. According to [10], when performing an exact quantum analysis of this theory (without the additional singlets which we have here), it has an IR description in terms of mesons $m_i^{\tilde{i}}$ and baryons B and $\tilde{B}$, constrained by $\det(m) - B\tilde{B} = \Lambda^{2N_f}$. With the additional singlets we, therefore, find that our $SU(N_f)$ gauge theory is equivalent to the theory of mesons and baryons with this constraint and with an additional superpotential of the form $M_{\tilde{i}}^i m_i^{\tilde{i}}$ (M denotes the mesons which couple, in the "magnetic" theory, to the

quarks which are charged under the local $SU(N_f)$, as described above). Integrating out the field M we find that $m = 0$. Hence, we are left just with the fields B and $\tilde{B}$ constrained by $B\tilde{B} = -\Lambda^{2N_f}$. These fields are charged under the local $U(1)$ symmetry, with opposite charges. We claim that this theory, when combined with the $SU(N_f - N_c)$ "magnetic" theory and with a $U(1)$ gauge symmetry coupling the two, is equivalent to the "electric" $SU(N_c)$ gauge theory. To form a gauge–invariant operator in the "magnetic" theory, we must take an operator in the $SU(N_f - N_c)$ "magnetic" theory (these are known to be in a one-to-one correspondence with the operators of the "electric" theory [9]) and add to it an appropriate power of B or $\tilde{B}$ so that the result is invariant under the local $U(1)$ symmetry. Obviously there is just one way to do this. Thus, the resulting theories along these flat directions are also dual. A similar analysis holds for the case $\tilde{k} = 0$ when $N_f < N_c$.

The other cases we had problems with were those of $k < N_c - N_f$ when $N_f < N_c$, and those of $\tilde{k} < N_f - N_c$ when $N_f > N_c$. In all of these cases one of the gauge theories we find along this flat direction has a number of flavors smaller than the number of colors. In this case it is known that SQCD does not have a stable vacuum, due to quantum effects. Thus, it is possible that these flat directions may not even exist in the full quantum theory, due to the quantum effects that we have ignored. In any case we expect the quantum corrections to be important along these flat directions and we shall not analyze them here any further. Except for these cases, all flat directions in the "electric" theory are identified with flat directions in the corresponding "magnetic" theory. In all cases we flow to theories which are already known to be dual, either by the description of SQCD for $N_f = N_c$ and $N_f = N_c + 1$ in terms of bound states [10], or by the duality transformation of SQCD [9].

The case of $\tilde{N}_f > 0$ is similar to the previous case. However, according to (2.6), in this case the mass operator $m\text{Tr}\,(X^2)$ is identified with a non–trivial linear combination of $\text{Tr}\,(Y^2)$ and $z_j \tilde{z}^j$:

$$m\text{Tr}\,(X^2) \sim m(\tilde{a}\text{Tr}\,(Y^2) + \tilde{b}z_j\tilde{z}^j). \tag{3.16}$$

In the "electric" theory, the analysis of the VEVs of X is the same as in the previous case (assuming that the Z-quarks do not get VEVs). The VEV of X gives a mass to all Z-quarks (except in the case $k = 0$), and, therefore, we flow to the same theories we discussed above (for $k > 0$). In the "magnetic" theory, we must now find VEVs for Y and for the z-quarks which will solve the equations of motion coming from the superpotential perturbed by the terms $\text{Tr}\,(Y^2)$ and $z_j \tilde{z}^j$. The solutions for which the z-quarks do not get VEVs are dual to solutions in which the Z-quarks do get VEVs. Hence, we will analyze here the case in which

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all z-quarks get VEVs. These VEVs must be orthogonal and of equal magnitude for all the z-quarks and anti–quarks (i.e. they have the form $z_j^a = A\delta_j^a$, $\tilde{z}_a^j = A\delta_a^j$). In this case we find that we flow exactly to the theories described above. The VEVs of z, $\tilde{z}$ and Y break the gauge symmetry to $SU(\tilde{k}) \times SU(2N_f - N_c - \tilde{k}) \times U(1)$. All the z-quarks and components of Y either become massive or are swallowed by the Higgs mechanism. The resulting theories are exactly the theories discussed above, for which we already investigated the duality. The cases in which the Z-quarks get VEVs (and the z-quarks do not), and the cases in which we add a term $\mathrm{Tr}\ (Y^2)$ instead of $\mathrm{Tr}\ (X^2)$ are all analogous. Note that we assumed here that $N_f > N_c/2$. If this does not hold, we expect the quantum corrections always to be important, since we always seem to flow to a SQCD theory with a number of flavors smaller than the number of colors.

We must still discuss what happens when $k = 0$ and $\tilde{N}_f > 0$. In this case in the "electric" theory the Z-quarks do not become massive, and integrating out the field X generates a (non–renormalizable) interaction between them of the form $W = Z_a^j \tilde{Z}_k^a Z_b^k \tilde{Z}_j^b$, which forces their R charge to be $\frac{1}{2}$. Thus, we flow to a SQCD theory with N_f flavors of one type, $\tilde{N}_f$ flavors of another type (with $R = \frac{1}{2}$) and N_c colors. In the "magnetic" theory, along the corresponding flat direction, only Y gets a VEV, which breaks the gauge group to $SU(N_f) \times SU(N_f + \tilde{N}_f - N_c) \times U(1)$. The mass of the z-quarks gets contributions both from the explicit mass term (3.16) and from the VEV of Y through the superpotential. For the duality to hold, these terms must cancel out for the z-quarks which are charged under the $SU(N_f + \tilde{N}_f - N_c)$ gauge group. Using the values of $\tilde{a}$ and $\tilde{b}$ which we found above we see that this indeed occurs. The "magnetic" theory turns out to be a "sum" of two theories. The first is a "magnetic" theory of N_f flavors and N_f colors (which behaves as in the discussion above). The second is a theory of N_f "normal" flavors and $\tilde{N}_f$ additional flavors, which have a superpotential analogous to the one we have found in the "electric" theory, with gauge group $SU(N_f + \tilde{N}_f - N_c)$. We claim that this theory is dual to the theory which we have found when we considered the corresponding flow from the "electric" theory. It includes couplings of the form $Mq\tilde{q} + Mq\tilde{z} + Mz\tilde{q}$, but not $Mz\tilde{z}$, where the mesons may be identified with the operators $Q\tilde{Q}$, $Z\tilde{Q}$ and $Q\tilde{Z}$ in the "electric" theory. This duality satisfies the usual requirements, i.e. the 't Hooft anomaly conditions for the global symmetry group (which is the same as the original global symmetry group) and the identification of the bound states. In fact, we can flow to this duality also directly from Seiberg's duality [9], starting from a SQCD theory with $N_f + \tilde{N}_f$ flavors and N_c colors and perturbing it by an operator of the form $M_k^j M_j^k$, where we include all mesons composed of the last $\tilde{N}_f$ flavors. Thus, we find that also this case is compatible with the duality.

4. Flat directions in the dual theories

In this section we will analyze some of the flat directions of the dual theories, verifying that their behavior is consistent with the duality. Since we do not analyze here the quantum corrections to the superpotentials of the two theories, we cannot prove that these flat directions indeed exist and behave as we describe. However, all the terms one can imagine in the superpotential do not seem to eliminate these flat directions of the classical theory. We shall, therefore, assume their existence in the full theory. The consistency of our results strongly suggests that these flat directions do indeed exist. In this section we study, for simplicity, only flat directions along which just one gauge–invariant operator gets a VEV while all others do not. We find that along some flat directions the theory flows to another dual theory, along some other flat directions the "electric" and "magnetic" theories both flow to the same theory in the IR, and along one flat direction the duality flows to Seiberg's duality [9]. The discussion of the flat direction for which the meson operator $M_{\tilde{i}}^i$ gets a VEV is more complicated and, therefore, is postponed to the next section.

4.1. Mesonic flat directions

In Seiberg's duality [9], the effect of giving a meson a VEV on the "electric" ("magnetic") theory is equivalent to the effect of adding a mass term (proportional to the meson operator) on the "magnetic" ("electric") theory. For the dual theories we analyze, it can easily be seen that giving the meson $M_{\tilde{i}}^i$ a VEV is similar to adding a term proportional to $N_{\tilde{i}}^i$ to the superpotential. Similarly, giving the generalized meson $N_{\tilde{i}}^i$ a VEV is dual in this sense to adding a perturbation proportional to $M_{\tilde{i}}^i$ (i.e. a mass term in the "electric" theory). Thus, the analysis of the flat direction along which only $N_{\tilde{i}}^i$ gets a VEV follows the analysis of the Q-quark mass perturbation described in the previous section.

Let us give a VEV only to the generalized meson $Q^1 X \tilde{Q}_1 \sim N_1^1$, keeping the VEVs of all other gauge–invariant operators zero. In the "electric" theory, this corresponds (up to color transformations) to Q_1^1, $\tilde{Q}_1^2$ and X_2^1 getting VEVs, which can be checked to be a flat direction of the theory. The color group breaks to $SU(N_c - 2)$, with $N_f - 1$ flavors remaining massless. Some components of X also get a mass from the superpotential, leaving the adjoint of $SU(N_c - 2)$ massless. Thus, along this flat direction we flow to the "electric" theory with $N_f - 1$ flavors and $N_c - 2$ colors. $\tilde{N}_f$ remains unchanged along this flat direction. In the "magnetic" theory, along this flat direction only N_1^1 gets a VEV. This leads to a mass term for q_1 and $\tilde{q}^1$, and we flow to the "magnetic"

theory with $(N_f - 1)$ q-flavors, $2N_f + \tilde{N}_f - N_c$ colors, and with $\tilde{N}_f$ unchanged. Since $2N_f + \tilde{N}_f - N_c = 2(N_f - 1) + \tilde{N}_f - (N_c - 2)$ this is exactly the dual to the "electric" theory which we found above. We conclude that the duality holds along this flat direction.

Another flat direction of these theories is realized by giving a VEV to $Z^j \tilde{Q}_{\tilde{i}} \sim M_{\tilde{i}}^j$. The analysis of this flat direction is similar to the analysis of the perturbation by this operator carried out in the previous section. We should just interchange the "electric" and "magnetic" theories. In the "electric" theory, $\tilde{N}_f$ decreases by one and the gauge group breaks to $SU(N_c - 1)$, while in the "magnetic" theory, $\tilde{N}_f$ decreases by one without a change in the gauge group. Again, the duality is preserved.

Yet another flat direction is obtained by giving a VEV to $Z^2 \tilde{Z}_1 \sim z_1 \tilde{z}^2$. In both theories, this reduces the number of colors by one. The quarks $Z^2, \tilde{Z}_1, z_1$ and $\tilde{z}^2$ are all swallowed by the Higgs mechanism, while the quarks $Z^1, \tilde{Z}_2, z_2$ and $\tilde{z}^1$, as well as some components of X and Y, get masses from the superpotential. Thus, in both theories $\tilde{N}_f$ is reduced by two while the number of colors is reduced by one. This result is compatible with the duality. The discussion of the flat direction along which a diagonal Z-meson gets a VEV is analogous to the discussion of the perturbation by the Z-mass operator in the previous section and, therefore, will not be repeated here.

4.2. The baryonic flat direction

In Seiberg's dual theories [9], the baryon composed of quarks is equivalent to the baryon composed of dual quarks. In [11] the flat direction along which this baryon gets a VEV was analyzed. It was found that the same IR theory results from both the "electric" and the "magnetic" theories. In the dual theories analyzed here we also expect the gauge groups to be completely broken along this flat direction, and to find the same IR effective theory resulting from both theories. However, in the present case the baryon which is composed of only quarks in one theory is equivalent to the baryon composed of quarks and adjoint fields in the dual theory. We shall analyze here the simplest case, in which the baryon made of only quarks (in one of the theories) gets a VEV. It seems quite obvious (and has been checked in several cases) that the behavior along flat directions for which other baryons (which are made in both theories out of the corresponding Q and Z quarks and adjoint fields) is similar. We shall find in this case that the "electric" and the "magnetic" theories indeed give rise to the same IR theory.

As described in section 2, the baryon operator in the "electric" theory is given by

$$B_{el}^{[i_1,\cdots,i_k][i_{k+1},\cdots,i_n][j_1,\cdots,j_m]} = \epsilon^{\alpha_1,\cdots,\alpha_{N_c}} X_{\alpha_1}^{\beta_1} \cdots X_{\alpha_k}^{\beta_k} Q_{\beta_1}^{i_1} \cdots Q_{\beta_k}^{i_k} Q_{\alpha_{k+1}}^{i_{k+1}} \cdots Q_{\alpha_n}^{i_n} \cdot$$
$$Z_{\alpha_{n+1}}^{j_1} \cdots Z_{\alpha_{n+m}}^{j_m} \tag{4.1}$$

with $m + n = N_c$, and in the "magnetic" theory by

$$B_{mag}^{[\tilde{i}_1,\cdots,\tilde{i}_{\tilde{k}}][\tilde{i}_{\tilde{k}+1},\cdots,\tilde{i}_{\tilde{n}}][\tilde{j}_1,\cdots,\tilde{j}_{\tilde{m}}]} = \epsilon_{\alpha_1,\cdots,\alpha_{2N_f+\tilde{N}_f-N_c}} Y_{\beta_1}^{\alpha_1} \cdots Y_{\beta_{\tilde{k}}}^{\alpha_{\tilde{k}}} q_{\tilde{i}_1}^{\beta_1} \cdots q_{\tilde{i}_{\tilde{k}}}^{\beta_{\tilde{k}}} q_{\tilde{i}_{\tilde{k}+1}}^{\alpha_{\tilde{k}+1}} \cdots q_{\tilde{i}_{\tilde{n}}}^{\alpha_{\tilde{n}}} \cdot$$
$$z_{\tilde{j}_1}^{\alpha_{\tilde{n}+1}} \cdots z_{\tilde{j}_{\tilde{m}}}^{\alpha_{\tilde{n}+\tilde{m}}} \tag{4.2}$$

with $\tilde{m} + \tilde{n} = 2N_f + \tilde{N}_f - N_c$. The identification between the operators in the two theories is by $\tilde{k} = N_f - n + k$, $\tilde{n} = 2N_f - n$, $\tilde{m} = \tilde{N}_f - m$, and by $\epsilon^{i_1,\cdots,i_k,\tilde{i}_{\tilde{k}+1},\cdots,\tilde{i}_{\tilde{n}}} = \epsilon^{\tilde{i}_1,\cdots,\tilde{i}_{\tilde{k}},i_{k+1},\cdots,i_n} = \epsilon^{j_1,\cdots,j_m,\tilde{j}_1,\cdots,\tilde{j}_{\tilde{m}}} = 1$.

Let us choose the baryon made of the first N_c Q-quarks, $B_{el}^{[\,][1,\cdots,N_c][\,]}$, to get a non–zero VEV. This baryon only exists for $N_f \geq N_c$ which we will assume in this subsection. In the "magnetic" theory this baryon is equivalent to $B_{mag}^{[N_c+1,\cdots,N_f][1,\cdots,N_f][1,\cdots,\tilde{N}_f]}$. In the "electric" theory, the first N_c Q-quarks get VEVs and are swallowed by the Higgs mechanism, with the gauge group being completely broken (one chiral superfield remains massless, labeling the flat direction. We shall ignore it since it is a non–interacting singlet). All other fields remain massless. As in Seiberg's theory [11], the global symmetry breaks to

$$SU(N_c) \times SU(N_f - N_c) \times SU(N_f) \times SU(\tilde{N}_f) \times U(1)_{\hat{B}} \times U(1)_{\hat{Z}} \times U(1)_{\hat{R}}, \tag{4.3}$$

and the charges of the massless fields turn out to be :

$Q_a^{\hat{i}}$ (	N_c,	$N_f - N_c$,	1,	1,	$\frac{N_f}{(N_f-N_c)N_c}$,	$-\frac{N_f}{N_c}$,	$\frac{3N_f+\tilde{N}_f-2N_c}{3(N_f-N_c)}$)
$\tilde{Q}_{\tilde{i}}^a$ (	$\overline{N_c}$,	1,	$\overline{N_f}$,	1,	$-\frac{1}{N_c}$,	$\frac{N_f-N_c}{N_c}$,	$1 + \frac{1}{3}\frac{\tilde{N}_f-2N_c}{N_f}$)
Z_a^j (	N_c,	1,	1,	$\tilde{N}_f$,	$\frac{1}{N_c}$,	$-\frac{(N_c+\tilde{N}_f)N_f}{N_c\tilde{N}_f}$,	$\frac{2}{3}$)
$\tilde{Z}_j^a$ (	$\overline{N_c}$,	1,	1,	$\overline{\tilde{N}_f}$,	$-\frac{1}{N_c}$,	$\frac{(N_c+\tilde{N}_f)N_f}{N_c\tilde{N}_f}$,	$\frac{2}{3}$)
X_b^a (	$N_c^2 - 1$,	1,	1,	1,	0,	0,	$\frac{2}{3}$).

The superpotential associated with these massless fields is (assuming no quantum corrections) the same as the original superpotential (2.1), since all the fields involved remain massless. This is the only classical interaction between the fields in the IR.

In the "magnetic" theory it is obvious from the form of the baryon operator that all $N_f + \tilde{N}_f$ quarks must get a VEV. This is, of course, possible since $N_f \geq N_c$ implies $N_f + \tilde{N}_f \leq 2N_f + \tilde{N}_f - N_c$. We can always choose (by flavor and color transformations) the VEV of q to be diagonal, $q_j^i = \delta_j^i A_i$. The F-term equations force us to choose the non–zero components of z (by flavor and color transformations) to be of the form $z_j^{N_f+j} = z_j$ for $j = 1, \cdots, \tilde{N}_f$. Then, the simplest choice for the VEV of Y leading to a non–zero baryon is $Y_{N_c+i}^{N_f + \tilde{N}_f + i} = y_i$ for $i = 1, \cdots, N_f - N_c$ with all other components of Y vanishing. The gauge group is completely broken (the q and z VEVs break it to $SU(N_f - N_c)$, which the Y VEV then breaks completely), and the D terms in the scalar potential lead to constraints which determine (up to an overall constant) the VEVs A_i, z_i and y_i. The solution turns out to be $A_i = A$ for $i = 1, \cdots, N_c$, $A_i = B$ for $i = N_c+1, \cdots, N_f$, $z_i = z$ for $i = 1, \cdots, \tilde{N}_f$ and $y_i = y$ for $i = 1, \cdots, N_f - N_c$, with three linear equations relating $|A|^2, |B|^2, |z|^2$ and $|y|^2$. Thus, we have found the VEVs of all fields along this flat direction (all other VEVs vanish).

The VEV of q breaks the q-flavor $SU(N_f)$ and color $SU(2N_f + \tilde{N}_f - N_c)$ symmetries together to $SU(N_c) \times SU(N_f - N_c) \times SU(N_f + \tilde{N}_f - N_c)$ where the first two factors are global symmetries which are diagonal products of the flavor and color symmetries, and the last factor is a local symmetry. The VEV of z breaks the local $SU(N_f + \tilde{N}_f - N_c)$ symmetry to a global $SU(\tilde{N}_f)$ (which is a diagonal combination of the original $SU(\tilde{N}_f)$ flavor group and part of the color group) and a local $SU(N_f - N_c)$. The Y VEV breaks the $SU(N_f - N_c) \times SU(N_f - N_c)$ symmetry to its diagonal subgroup which remains a global symmetry. The $SU(N_f)$ $\tilde{q}$-flavor symmetry still remains, and thus we are left with the same global symmetry as we found for the "electric" theory (4.3).

Next, we should check which fields become massive along this flat direction. All of the terms in the superpotential (2.3) directly give a mass to some of the fields when plugging in the VEVs. To analyze the effect of these terms on Y it is simplest to decompose the matrix Y as follows :

$$Y = \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 \\ Y_5 & Y_6 & Y_7 & Y_8 \\ Y_9 & Y_{10} & Y_{11} & Y_{12} \\ Y_{13} & Y_{14} & Y_{15} & Y_{16} \end{pmatrix} \tag{4.4}$$

where the $2N_f + \tilde{N}_f - N_c$ rows and columns of the matrix Y are decomposed into groups of size $(N_c, N_f - N_c, \tilde{N}_f, N_f - N_c)$. We chose a VEV in which Y_8 is diagonal and the VEVs of all the other submatrices vanish. It is then easy to see that the fields which get a mass from the superpotential are all the components of $\tilde{q}$ and $\tilde{z}$; $M_{\tilde{i}}^i$ for $i = N_c + 1, \cdots, N_f$;

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$N_{\tilde{i}}^i$; $M_{\tilde{i}}^j$; $M_{\tilde{i}}^i$ for $i = N_c + 1, \cdots, N_f$; Y_2; $Y_6 + Y_{16}$; a linear combination of Y_9 and M_j^i for $i = 1, \cdots, N_c$; Y_{10}; Y_{11}; Y_{12}; Y_{13}; Y_{14}; and Y_{15}. Some of the fields which remain massless $((2N_f + \tilde{N}_f - N_c)^2 - 1$ of them) are swallowed by the Higgs mechanism and, as in the "electric" theory, one field remains massless and labels the flat direction. These can be seen to be all the components of q (including $N_f(2N_f + \tilde{N}_f - N_c)$ superfields), all the components of z (including $\tilde{N}_f(2N_f + \tilde{N}_f - N_c)$ superfields) and the matrices $Y_5, Y_6 - Y_{16}, Y_7$ and Y_8, which include $(N_f - N_c)(2N_f + \tilde{N}_f - N_c)$ more superfields.

The fields which remain massless are thus Y_4; M_i^i for $i = 1, \cdots, N_c$; Y_3; a linear combination of Y_9 and M_j^i for $i = 1, \cdots, N_c$; and Y_1. By computing the global quantum numbers of all these fields, one can find that they are exactly the same as those we found above for the "electric" fields, if we identify them (in the order in which they appear above) with the "electric" fields which remained massless (in the order they appear in the table which gives their quantum numbers). The superpotential of the massless fields turns out to be, after integrating out the massive fields, exactly the same as the one we have found in the "electric" theory, with the appropriate identification of the fields. The only contribution is from the Tr (Y^3) term acting on the submatrix

$$\hat{Y} = \begin{pmatrix} Y_1 & Y_3 \\ Y_9 & Y_{11} \end{pmatrix}. \tag{4.5}$$

Thus, we find, as in Seiberg's duality, that along this baryonic flat direction the "electric" and "magnetic" theories go over to the same IR theory, providing additional support for the duality conjecture.

4.3. The adjoint flat direction – flow to Seiberg's duality

Another field which can sometimes get a VEV along a flat direction is the field Tr (X^2) (or Tr (Y^2) in the "magnetic" theory). Since Tr $(X) = 0$ and the equation of motion of X (in the absence of VEVs for the Z-quarks) forces X^2 to be proportional to the identity matrix, this is only possible when N_c is even. In this case we may have a flat direction for which $X_i^i = A$ $(i = 1, \cdots, N_c/2)$ and $X_i^i = -A$ $(i = N_c/2 + 1, \cdots, N_c)$. Along this flat direction the gauge group breaks to $SU(N_c/2) \times SU(N_c/2) \times U(1)$, and the Z-quarks all become massive. Thus, we flow to two SQCD theories with N_f flavors and $N_c/2$ colors, coupled by a $U(1)$ gauge symmetry. The fields corresponding to the diagonal submatrices of X become massive, while those corresponding to the off–diagonal submatrices of X remain massless and are swallowed by the Higgs mechanism.

In the "magnetic" theory, according to (2.6), Y, z, and $\tilde{z}$ must all get VEVs along this flat direction (since the operator identified with $\text{Tr}\,(X^2)$ must be non–zero while the operator identified with $Z^j \tilde{Z}_j$ must be zero). In fact, z and $\tilde{z}$ must get diagonal VEVs, because we assume that the adjoint z-mesons do not get VEVs. Examining the equations of motion and the tracelessness condition of Y we find that this is only possible for even $2N_f - N_c$. Then, Y can get a VEV of the form $Y_i^i = A$ ($i = 1, \cdots, N_f - N_c/2$) and $Y_i^i = -A$ ($i = N_f - N_c/2 + 1, \cdots, 2N_f - N_c$), where A is proportional to the VEV of z and $\tilde{z}$. The gauge group breaks to $SU(N_f - N_c/2) \times SU(N_f - N_c/2) \times U(1)$, and the z-quarks are swallowed by the Higgs mechanism (or get a mass from the superpotential). We, therefore, flow to two "magnetic" SQCD theories with N_f flavors and $N_f - N_c/2$ colors. The fields which were contained in Y either get a mass from the superpotential or remain massless and are swallowed by the Higgs mechanism. There is also an additional $U(1)$ local symmetry which couples the two theories. The superpotentials in the "magnetic" theories flow to two copies of the superpotential of Seiberg's "magnetic" theory in the same way as described in the discussion of the adjoint mass perturbation in section 3.3.

Thus, we find here the same results as those that we obtained in the investigation of the adjoint mass perturbation (in the cases for which the quantum corrections were not important). Again, we flow to two copies of Seiberg's dual theories [9], this time with the same gauge group for the two copies, and with an additional $U(1)$ gauge symmetry in both theories. As in the analysis of the adjoint mass perturbation, we expect quantum effects to be important when $N_f \leq N_c/2$. The above analysis is only relevant when $N_f > N_c/2$.

5. The mesonic flat direction

Unlike the previous flat directions, we can prove that a flat direction along which the meson operator gets a VEV exists in the quantum theory. This may be done as for ordinary SQCD, by giving masses to all the fields, computing the gluino condensate in the low–energy Yang–Mills theory, and relating it by the Konishi anomaly to the meson VEV. This gives the VEV of the meson in the massive theory. By taking the masses to zero in various ways, we can get any meson VEV of rank smaller than N_c. We cannot prove in this way that no other fields get VEVs along this flat direction, but this is a reasonable assumption, which we will justify by discussing the possible corrections to the superpotential in the two theories. Although, for simplicity, we shall limit the discussion of this section to the case $\tilde{N}_f = 0$ corresponding to [14], it can be generalized to arbitrary $\tilde{N}_f$.

5.1. The mesonic flat direction in the "electric" theory

The classical superpotential in the "electric" theory of $\tilde{N}_f = 0$ is of the form

$$W_{el} = \text{Tr } (X^3). \tag{5.1}$$

In the quantum theory, there are two types of operators which may appear in the superpotential without breaking any of the symmetries of the theory. The first type of operators involves the quark fields. In order to respect the flavor symmetries these can be of the form $\det(QX^n\tilde{Q})$ for some integer n (which must be larger than zero if $N_f \geq N_c$, and non–negative otherwise). The second type of operators involves only the field X, like $\text{Tr } (X^n)$ (there is a finite number of independent operators of this sort) or $\det(X)$. All these operators must of course appear with appropriate powers so that the resulting R charge will be 2. Of course, products of operators of both types may also appear. Envisioning the most general superpotential which one can write down using these terms, it is hard to see how $X = 0$ will not turn out to be a solution of the equations of motion also when we give a VEV to one quark and one anti–quark flavor (so as to be along the mesonic flat direction). We assume, of course, that the point for which all VEVs vanish is in the quantum moduli space of the theory for all $N_f > N_c/2$. Otherwise the discussion of the duality should be changed completely. Of course, in the "electric" theory, the rank of the meson VEV cannot be larger than N_c. Hence, it is reasonable to assume that the quantum theory has a mesonic flat direction, for which the meson is the only gauge–invariant operator getting a VEV, at least for mesons of small rank.

Along this flat direction (assuming a meson VEV of rank one), since one quark and one anti–quark acquire a VEV and X does not, the gauge symmetry is broken to $SU(N_c - 1)$. We will choose the fields which acquire non–zero VEVs to be $Q_{N_c}^{N_f}$ and $\tilde{Q}_{N_f}^{N_c}$. The fields Q^{N_f} and $\tilde{Q}_{N_f}$ (all components apart from one which labels the flat direction) now get swallowed by the Higgs mechanism. The field X includes an adjoint, a fundamental, an anti–fundamental and a singlet representation of the new gauge group. Ignoring the quantum corrections all of these fields also remain massless. Thus, we obviously do not flow to the "electric" theory with $N_f - 1$ flavors and $N_c - 1$ colors, since we have additional superfields which have non–trivial interactions (classically among themselves and with the field X, and in the quantum theory perhaps with the quarks as well). Denoting the $SU(N_c - 1)$ adjoint field in X by $\hat{X}$, the fundamental and anti–fundamental fields by Z

and $\tilde{Z}$, and the singlet field (with an arbitrary normalization) by A, one can easily find that the original superpotential becomes

$$W = \text{Tr}\,(X^3)$$
$$= \text{Tr}\,(\hat{X}^3) + 3A\text{Tr}\,(\hat{X}^2) + 3Z_a\hat{X}^a_b\tilde{Z}^b - 3(N_c - 2)AZ_a\tilde{Z}^a - (N_c - 2)(N_c^2 - N_c + 1)A^3. \tag{5.2}$$

The resulting theory has a global symmetry

$$SU(N_f - 1) \times SU(N_f - 1) \times U(1)_{\hat{B}} \times U(1)_Z \times U(1)_{\hat{R}}, \tag{5.3}$$

where $U(1)_{\hat{B}}$ and $U(1)_{\hat{R}}$ are the new baryon number and R-charge symmetries (the old ones were broken by the VEV). There is an additional $U(1)_Z$ which remains a symmetry of the theory. It is a combination of two $U(1)$'s originating from the two (broken) flavor $SU(N_f)$ groups and of a $U(1)$ originating from the (broken) color group.

In addition to the previously mentioned fields, the quarks (and anti–quarks) whose color index corresponds to the direction of the VEVs also remain massless and decouple from the theory. The full field content of the resulting theory, with the quantum numbers of $SU(N_c - 1)$ and of the global symmetry group $SU(N_f - 1) \times SU(N_f - 1) \times U(1)_{\hat{B}} \times U(1)_Z \times U(1)_{\hat{R}}$, is :

Q^i_a	$N_c - 1$	($N_f - 1$,	1,	$\frac{1}{N_c-1}$,	$\frac{N_c-N_f}{N_c-1}$,	$\frac{N_f - \frac{2}{3}N_c}{N_f-1}$)
$Q^i_{N_c}$	1	($N_f - 1$,	1,	0,	N_f,	$\frac{N_f - \frac{2}{3}N_c}{N_f-1}$)
$\tilde{Q}^a_{\tilde{i}}$	$\overline{N_c - 1}$	(1,	$\overline{N_f - 1}$,	$-\frac{1}{N_c-1}$,	$\frac{N_f-N_c}{N_c-1}$,	$\frac{N_f - \frac{2}{3}N_c}{N_f-1}$)
$\tilde{Q}^{N_c}_{\tilde{i}}$	1	(1,	$\overline{N_f - 1}$,	0,	$-N_f$,	$\frac{N_f - \frac{2}{3}N_c}{N_f-1}$)
$\hat{X}^a_b$	$((N_c - 1)^2 - 1)$	(1,	1,	0,	0,	$\frac{2}{3}$)
Z_a	$N_c - 1$	(1,	1,	$\frac{1}{N_c-1}$,	$-\frac{N_c(N_f-1)}{N_c-1}$,	$\frac{2}{3}$)
$\tilde{Z}^a$	$\overline{N_c - 1}$	(1,	1,	$-\frac{1}{N_c-1}$,	$\frac{N_c(N_f-1)}{N_c-1}$,	$\frac{2}{3}$)
A	1	(1,	1,	0,	0,	$\frac{2}{3}$)
W_α	$((N_c - 1)^2 - 1)$	(1,	1,	0,	0,	1).

The R charge pertains to the lowest component of each superfield, and the superpotential is given by (5.2).

5.2. The mesonic flat direction in the "magnetic" theory

In the "magnetic" theory of $\tilde{N}_f = 0$ [14] the classical superpotential is of the form

$$W_{mag} = M_{\tilde{i}}^i q_i Y \tilde{q}^{\tilde{i}} + N_{\tilde{i}}^i q_i \tilde{q}^{\tilde{i}} + \text{Tr}\,(Y^3). \qquad (5.4)$$

For this superpotential it is obvious that any meson VEV, with zero VEVs for the other fields, is a flat direction. As in Seiberg's "magnetic" theory [9], this superpotential (at least for $N_f > N_c$) is subject to quantum corrections. These are necessary in order to match the "electric" theory, in which a meson cannot get a VEV of rank higher than N_c. Possible terms in the full quantum superpotential which include meson operators are for instance $\det(M)$ or $MqY^n\tilde{q}$, raised to appropriate powers so as to have the correct R charge. However, it seems that any meson VEV of small rank, with zero VEVs for all other fields, must be a solution of the equations of motion for the most general superpotential which one can write. For instance, the equation of motion of the possible term in the superpotential of the form $\det(M)^{1/(N_f - \frac{2}{3}N_c)}$ vanishes whenever the rank of M is less than $\frac{2}{3}N_c$. Thus, it seems (though again we have no rigorous proof of this) that also in the "magnetic" theory there is a flat direction, for which M is non–zero of small rank (in particular of rank 1 as we want) while all other operators vanish.

The effect of going far along this flat direction is less clear than in the "electric" theory. In Seiberg's "magnetic" theory [9], a VEV to M was equivalent to a mass term for one of the dual quarks. Thus, going along this flat direction we integrated out the quark that became massive. However, here M is the coefficient of $qY\tilde{q}$, so it is not clear what is the meaning of taking one component of M to be very large. The problem arises because the relevant term in the superpotential is non–renormalizable. We want to take $M_{N_f}^{N_f}$ to be very large, and, therefore, the high–energy action which leads to the generation of this non–renormalizable term becomes important. We will proceed by choosing a particular high–energy renormalizable action which leads to (5.4) upon integrating out the massive fields. The choice is not unique, but our choice seems to be the simplest possible one. We shall later be able to check that the resulting low–energy theory is indeed dual to the theory we found along this flat direction in the "electric" theory.

To define the high–energy theory we introduce new massive superfields $\hat{W}_a^{\tilde{i}}$, $W_{\tilde{i}}^a$, $\hat{\tilde{W}}_i^b$ and $\tilde{W}_b^i$, where i and $\tilde{i}$ are flavor indices of the $SU(N_f)$ flavor groups and a and b are color indices of $SU(2N_f - N_c)$. We then write the superpotential

$$W_{UV} = M_{\tilde{i}}^i q_i^a \hat{W}_a^{\tilde{i}} + Y_a^b \tilde{q}_b^{\tilde{i}} W_{\tilde{i}}^a + M_W \hat{W}_a^{\tilde{i}} W_{\tilde{i}}^a + M_{\tilde{i}}^i \tilde{q}_b^{\tilde{i}} \hat{\tilde{W}}_i^b + Y_a^b q_i^a \tilde{W}_b^i + M_W \hat{\tilde{W}}_i^b \tilde{W}_b^i. \qquad (5.5)$$

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This superpotential does not break any of the symmetries of the theory (upon giving the W fields appropriate charges). Upon integrating out the massive W fields it leads to the term $M_{\tilde{i}}^{i} q_{i}^{a} Y_{a}^{b} \tilde{q}_{b}^{\tilde{i}}$ in the low–energy superpotential. Now, in this high–energy theory, a non–zero $M_{N_{f}}^{N_{f}}$ is still a flat direction of the theory, and we can analyze what happens when we take it to be large (of the same order of magnitude as M_{W}). We find that the field proportional to $M_{W} W_{N_{f}} + M_{N_{f}}^{N_{f}} q_{N_{f}}$ gets a mass instead of $W_{N_{f}}$, as does the field proportional to $M_{W} \tilde{W}^{N_{f}} + M_{N_{f}}^{N_{f}} \tilde{q}^{N_{f}}$, while the orthogonal combinations remain massless. We will denote these massless fields by z^{a} and $\tilde{z}_{a}$, and up to a normalization constant they are given by

$$
\begin{aligned}
z^{a} &= -M_{N_{f}}^{N_{f}} W_{N_{f}}^{a} + M_{W} q_{N_{f}}^{a} \\
\tilde{z}_{a} &= -M_{N_{f}}^{N_{f}} \tilde{W}_{a}^{N_{f}} + M_{W} \tilde{q}_{a}^{N_{f}}.
\end{aligned}
\tag{5.6}
$$

We can now integrate out all the massive fields (using the full superpotential which includes also the other terms in (5.4)), and find the superpotential for the massless fields. This superpotential includes terms like (5.4) but involving only the first $N_{f} - 1$ flavors. In addition to this there are various terms involving the fields of the N_{f}'th flavor. The renormalizable terms are proportional to $z^{a} Y_{a}^{b} \tilde{z}_{b}$, $N_{\tilde{i}}^{N_{f}} \tilde{q}_{a}^{\tilde{i}} z^{a}$, $N_{N_{f}}^{i} q_{i}^{a} \tilde{z}_{a}$ and $N_{N_{f}}^{N_{f}} z^{a} \tilde{z}_{a}$, and there are additional non–renormalizable terms, involving also the fields $M_{N_{f}}^{i}$ and $M_{\tilde{i}}^{N_{f}}$. Similar results may be found from other high–energy superpotentials which give the same low–energy superpotential. The differences are generally only in irrelevant non–renormalizable terms. $M_{N_{f}}^{N_{f}}$ is also massless and labels the flat direction, as does one of the components of $Q^{N_{f}}$ and $\tilde{Q}_{N_{f}}$ in the "electric" theory.

As in the "electric" theory, we find that the remaining global symmetry is of the form $SU(N_{f} - 1) \times SU(N_{f} - 1) \times U(1)_{\hat{B}} \times U(1)_{Z} \times U(1)_{\hat{R}}$. The $U(1)_{Z}$ symmetry here is the sum of two $U(1)$ factors from the (broken) $SU(N_{f})$ flavor groups. The quantum numbers of the remaining massless fields under the (unbroken) $SU(2N_{f} - N_{c})$ color group and under the flavor group can easily be seen to be (where we choose the normalization of the new $U(1)_{Z}$ to agree with its normalization in the "electric" theory) :

q_i^a	$2N_f - N_c$	($\overline{N_f-1}$,	1,	$\frac{1}{2N_f-N_c}$,	-1,	$1-\frac{1}{3}\frac{4N_f-2N_c-1}{N_f-1}$)
$\tilde{q}_a^{\tilde{i}}$	$\overline{2N_f-N_c}$	(1,	N_f-1,	$-\frac{1}{2N_f-N_c}$,	1,	$1-\frac{1}{3}\frac{4N_f-2N_c-1}{N_f-1}$)
Y	$((2N_f-N_c)^2-1)$	(1,	1,	0,	0,	$\frac{2}{3}$)
$M_{\tilde{i}}^i$	1	(N_f-1,	$\overline{N_f-1}$,	0,	0,	$2\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$N_{\tilde{i}}^i$	1	(N_f-1,	$\overline{N_f-1}$,	0,	0,	$\frac{2}{3}+2\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$M_{\tilde{i}}^{N_f}$	1	(1,	$\overline{N_f-1}$,	0,	$-N_f$,	$\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$N_{\tilde{i}}^{N_f}$	1	(1,	$\overline{N_f-1}$,	0,	$-N_f$,	$\frac{2}{3}+\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$M_{N_f}^i$	1	(N_f-1,	1,	0,	N_f,	$\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$N_{N_f}^i$	1	(N_f-1,	1,	0,	N_f,	$\frac{2}{3}+\frac{N_f-\frac{2}{3}N_c}{N_f-1}$)
$N_{N_f}^{N_f}$	1	(1,	1,	0,	0,	$\frac{2}{3}$)
z^a	$2N_f-N_c$	(1,	1,	$\frac{1}{2N_f-N_c}$,	N_f-1,	$\frac{2}{3}$)
$\tilde{z}_a$	$\overline{2N_f-N_c}$	(1,	1,	$-\frac{1}{2N_f-N_c}$,	$-(N_f-1)$,	$\frac{2}{3}$)
W_α	$((2N_f-N_c)^2-1)$	(1,	1,	0,	0,	1).

5.3. Comparison of the resulting theories

Now that we have found the resulting theories along the mesonic flat direction, we can compare them to see that they are indeed dual. First, we notice that all singlets appearing in the "electric" theory have partners in the "magnetic" theory with the same quantum numbers, and, therefore, we can identify them. Next, we note that without the singlets, the "electric" theory is exactly the "electric" theory of section 2 with N_f-1 regular flavors, $\tilde{N}_f = 1$ and $N_c - 1$ colors. The "magnetic" theory is exactly the "magnetic" theory of section 2 with $N_f - 1$ regular flavors, $\tilde{N}_f = 1$ and $2N_f - N_c = 2(N_f - 1) + 1 - (N_c - 1)$ colors. Thus, these theories are dual to each other. By giving a mass to all the "electric" singlets we can flow directly to the dual theories described in section 2. In fact, by taking a meson VEV of rank k (with k equal non–zero eigenvalues) we can flow in this way to dual theories with $N_f - k$ regular quark flavors, $\tilde{N}_f = k$ and $N_c - k$ colors. Starting from a theory with general $N_f, \tilde{N}_f$ and N_c we would find that N_f and N_c decrease by k while $\tilde{N}_f$ increases by k, with no change in the "magnetic" gauge group, in a way consistent with the duality. All the duality theories we analyze may be obtained in this way from the $\tilde{N}_f = 0$ theory of [14], and this is in fact how we first discovered them.

6. Summary and conclusions

In this paper we described a duality transformation of $N = 1$ supersymmetric non–abelian gauge theories. This duality can be viewed as resulting from Kutasov's duality [14] by flowing along one of its flat directions. We have performed several non–trivial checks for the consistency of this duality. These include mass perturbations and the investigation of the theories along several flat directions.

To further analyze these theories one should understand the quantum corrections to their superpotential. This is necessary for the identification of all flat directions of the two theories. We recall that in some of the cases investigated in this work, and certainly in other cases as well, the quantum corrections are important. The quantum corrections may be similar in form to the quantum superpotential recently computed in [15]. It may also be interesting to generalize the duality transformation analyzed here to other gauge groups, such as $SO(N)$.

By flows along flat directions or mass perturbations we can connect our dual theories with those of Seiberg [9] and Kutasov [14]. Thus, it seems clear that the (as yet not well understood) mechanism behind the duality symmetry is the same for all of these cases. These include all cases which are known so far of $N = 1$ $SU(N)$ gauge theories which exhibit duality. Understanding this mechanism in one of the cases will, therefore, enable us to understand all of them. In particular, the case of $N_f = 0$ and $\tilde{N}_f = 2N_c$ seems to be related to the duality of $N = 2$ theories with $N_f = 2N_c$ [3-5]. From this theory we can flow to all other theories with $N_f = 0$, but not to theories with $N_f > 0$. The relation between the duality of section 2 and the $N = 2$ duality is under current investigation. Such a relation would also relate the $N = 2$ duality to the dualities of Seiberg [9] and Kutasov [14].

Another interesting generalization of our results is to duality symmetries between other $N = 1$ gauge theories, in particular theories with other matter representations. We have noticed that the 't-Hooft anomaly matching conditions associated with the original $SU(N_c)$ "electric" theory of SQCD are also satisfied by a class of "magnetic" theories, generalizing the solution found in [9]. The "magnetic" theories are $SU(\tilde{N}_c)$ gauge theories with $\tilde{N}_c = kN_f - N_c$, with k underlying mesons having $U(1)_R$ charges differing by multiples of 2 starting with $2 - 2\frac{N_c}{N_f}$ (for the scalar meson), and with the same dual quarks and gluinos. Moreover, we have found, for these cases, a complete identification between the gauge–invariant operators in the "electric" and "magnetic" theories. It would be interesting to

find out whether one can "elevate" these cases to a full duality symmetry. We hope to return to the various possible generalizations in future work. Clearly, the phenomenon of $N = 1$ duality is quite general. Hopefully, the investigation of these generalizations will shed light on the underlying mechanism behind the duality symmetry.

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